УДК 537.612.2, 537.612.3
DOI https://doi.org/10.32782/pet-2023-1-2

Volodymyr HOLOVATSKY
Doctor of Physical and Mathematical Sciences, Professor, Department of thermoelecrisity and medical physics, Chernivtsi National University, 2 Kotsyubynsky str., Chernivtsi, Ukraine, 58012
***ORCID ID**: https://orcid.org/0000-0002-5573-2562*
***SCOPUS-AUTHOR ID:** 6507899727*

Ihor HOLOVATSKYI
PhD student, Chernivtsi National University, 2 Kotsyubynsky str., Chernivtsi, Ukraine, 58012
***ORCID ID**: https://orcid.org/0000-0002-4435-4607*
***SCOPUS-AUTHOR ID:** 57202987360*

Yana HOLOVATSKA
Student, Computer Science major, New York University Abu Dhabi, UAE

Yaroslav STRUK
Candidate of Physical and Mathematical Sciences, Associate Professor at the Department of Information Technologies and Computer Physics, Chernivtsi National University, 2 Kotsyubynsky str., Chernivtsi, Ukraine, 58012
***SCOPUS-AUTHOR ID:** 54884488400*

To cite this article: Holovatsky V., Holovatskyi I., Holovatska Ya., Struk Ya. (2023). Rozrakhunok syly vzaiemodii tsylindrychnykh ta kiltsevykh mahnitiv [Oscillations of the resonant elastic pendulum]. *Physics and Educational Technology*, 1, 10–17, doi: https://doi.org/10.32782/pet-2023-1-2

OSCILLATIONS OF THE RESONANT ELASTIC PENDULUM

The purpose of this work is to investigate the peculiarities of oscillations of an elastic gravitational pendulum, which performs simultaneous coupled oscillations, like a spring and a mathematical pendulum, with a resonant frequency ratio of these oscillations (2:1). Determine the influence of initial conditions on the formation of stable periodic oscillation modes. Investigate the process of energy transfer from one subsystem to another.

On the basis of Lagrangian mechanics, neglecting the damping effect, the equations of motion of the pendulum were obtained and the numerical solutions were investigated using computer simulation. It was established that in such a system there is a periodic process of transfer of vibrational energy from one subsystem to another. It is shown that the amount of energy transferred and the period of this process depend on the initial conditions. The following initial conditions were found, under which complete energy transfer occurs, as well as conditions under which there is no influence of one subsystem on another.

The novelty of this work lies in the fact that for the first time a map of stable modes of oscillations is constructed, reflecting their evolution depending on the initial conditions of the oscillating system. Based on this oscillation map, it is possible to predict the main parameters of pendulum oscillations under arbitrary initial conditions. Visual and graphic interpretation of the solutions obtained in this work can be used in the study of other physical processes. The computer model is published and can be used in the educational process.

***Key words:** elastic pendulum, Lagrange equation, Lisage figures, resonant oscillations.*

Володимир ГОЛОВАЦЬКИЙ
доктор фізико-математичних наук, професор, професор кафедри термоелектрики та медичної фізики, Чернівецький національний університет імені Юрія Федьковича, вул. Коцюбинського 2, м. Чернівці, Україна, 58012
***ORCID ID**: https://orcid.org/0000-0002-5573-2562*
***SCOPUS-AUTHOR ID:** 6507899727*

Ігор ГОЛОВАЦЬКИЙ
аспірант кафедри інформаційних технологій та комп'ютерної фізики, Чернівецький національний університет імені Юрія Федьковича, вул. Коцюбинського 2, м. Чернівці, Україна, 58012
ORCID ID: *https://orcid.org/0000-0002-4435-4607*
SCOPUS-AUTHOR ID: *57202987360*

Яна ГОЛОВАЦЬКА
студентка спеціальності комп'ютерних наук Нью-Йоркського університету в Абу-Дабі, ОАЕ

Ярослав СТРУК
доцент кафедри інформаційних технологій та комп'ютерної фізики, Чернівецький національний університет імені Юрія Федьковича, вул. Коцюбинського 2, м. Чернівці, Україна, 58012
SCOPUS-AUTHOR ID: *54884488400*

Бібліографічний опис статті: Головацький В., Головацький І., Головацька Я., Струк Я. (2023). Розрахунок сили взаємодії циліндричних та кільцевих магнітів. *Фізика та освітні технології,* 1, 10–17, doi: https://doi.org/10.32782/pet-2023-1-2

РЕЗОНАНСНІ КОЛИВАННЯ ПРУЖНОГО ГРАВІТАЦІЙНОГО МАЯТНИКА

Мета даної роботи – дослідити особливості коливань пружного гравітаційного маятника, який здійснює одночасно зв'язані коливання, як пружинний та математичний маятники, при резонансному співвідношенні частот цих коливань (2:1). Визначити вплив початкових умов на утворення стабільних періодичних мод коливань. Дослідити процес передачі енергії з однієї підсистеми в іншу.

На основі механіки Лагранжа, нехтуючи ефектом затухання, отримано рівняння руху маятника та досліджено числові розв'язки за допомогою комп'ютерного моделювання. Встановлено, що у такій системі відбувається періодичний процес передачі енергії коливань з однієї підсистеми в іншу. Показано, що величина енергії, яка передається та період цього процесу залежить від початкових умов. Знайдено такі початкові умови, при яких відбувається повна передача енергії, а також умови, при яких відсутній вплив однієї підсистеми на іншу.

Новизна даної роботи полягає у тому, що вперше побудована карта стабільних мод коливань, що відображає їх еволюцію у залежності від початкових умов коливної системи. На основі цієї карти коливань можна спрогнозувати основні параметри коливань маятника при довільних початкових умовах. Наглядно-графічна інтерпретація розв'язків, отриманих у даній роботі, може використовуватись при вивченні інших фізичних процесів. Комп'ютерна модель опублікована з відкритим кодом і може використовуватись у навчальному процесі.

Ключові слова: *пружній гравітаційний маятник, рівняння Лагранжа, фігури Лісажу, резонансні коливання.*

1. Introduction

An elastic gravitational pendulum is an example of nonlinearly coupled oscillating systems. Resonant oscillations of this mechanical system are the subject of modern scientific researches because of their similarity to other complex oscillations that are often encountered in physics (Алдошин, Яковлев, 2016).

The first scientific study of an elastic gravitational pendulum was carried out in the work of Witt and Gorelik (Витт, Горелик, 1993). In this work, a system of Lagrange equations in a polar coordinate system was obtained, the analytical solutions of which are unknown. In the absence of the possibility of computer modeling, the authors performed analytical studies of the system of differential equations in the approximation of small fluctuations. The paper assumes the existence of resonant oscillations with periodic energy transfer from one subsystem to another. It was established that unlike linearly coupled oscillating systems (for example, two mathematical pendulums connected by a spring), where the depth and speed of energy transfer depends on the system parameters, in this problem these values are determined by the initial conditions.

After the first publication, studies of the oscillations of an elastic pendulum were carried out both theoretically (Olsson, 1976; Anisin, Davidovic, Babovic et al, 1993; Christensen, 2004; Lai, 1984; Carretero-Gonzalez, Nunez-Yepez, Salas-Brito, 1994; Sousa, Marcus, Caldas, 2018) and experimentally (Cross, 2017).

The analysis of the scientific literature showed that the conditions for the occurrence of certain modes of oscillation remain insufficiently researched, and the dependence of the period and magnitude of energy transfer on the initial conditions remains unknown.

In this paper on the basis of numerical solutions of the equation of motion [10], the peculiarities of the elastic pendulum oscillations were studied in the condition of resonance, when the frequency of the mathematical pendulum is twice less than the frequency of the spring pendulum. It is shown that the energy which is periodically transmitted from one oscillatory subsystem to another and the period of this process depend on the initial conditions. The dependence of the period of energy exchange between subsystems and its value on the energy of each subsystem is determined. The stable modes of oscillations and the features of their modification by changing the initial conditions are investigated. It is shown that for small oscillations, in the case when the energy of the gravitational pendulum is twice bigger than the energy of the spring pendulum, each mode of oscillation comes into a stable state in which there is no energy transmission between subsystems. On the coordinate plane of the initial conditions, an oscillation mode map is made, which demonstrates lines of several simple oscillation modes. The period of energy transmission between subsystems is constant along these lines.

2. Lagrangian and equations of motion

Consider a pendulum consisting of a mass m that is hanged on a spring (rubber thread). The length of unstretched spring is l_0, and k is its rigidity. The mass m satisfies the ratio

$$\frac{l}{g}=\frac{4m}{k} \qquad (1)$$

that provides the resonance ratio of the frequencies of the gravitational and spring pendulums. that provides the resonance ratio of the gravitational and spring pendulum frequencies. This expression is useless for the calculation of the required mass because rubber thread length depends on the m. Thus, $l=l_0+\Delta l$ and $\Delta l = mg/k$ from (1) we obtain a useful expression for the creation resonant pendulum.

$$\frac{\Delta l}{l_0}=\frac{l}{3}\,. \qquad (2)$$

The length of the spring r(t) and the angle of deviation of the pendulum from the equilibrium position φ(t) are chosen as generalized coordinates. The Lagrange function of the elastic pendulum is

$$L=\frac{m\dot{r}(t)^2}{2}+\frac{mr(t)^2\dot{\varphi}^2}{2}-\frac{k(r(t)-l_0)^2}{2}+mgr(t)\cos\varphi\,, \qquad (3)$$

where the first and second terms are the kinetic energy of the translational and rotational motions of the mass m, the third and fourth terms are the spring and the gravitational potential energies. The system of Lagrange equations is the following

$$\begin{cases} m\ddot{r}(t)-mr\varphi(t)^2+k\left(r(t)-l_0\right)-mg\cos(\varphi(t))=0 \\ r(t)\ddot{\varphi}(t)+2\dot{r}(t)\dot{\varphi}(t)+g\sin\varphi(t)=0 \end{cases} \qquad (4)$$

From equations (3) one can see, that gravitational and spring oscillations are coupled in a complicated way. Analytical solutions of the system of the differential equations (4) are unknown. Therefore, the numerical solutions of the problem are investigated in the next section.

3. Results of computer modelling

For the study of the resonant elastic pendulum oscillations, a computer model was created in the Wolfram Mathematica and published on the Wolfram Demonstrations Project website (Holovatsky, Holovatska, 2019). This program allows changing the initial conditions in a wide range and studying oscillations of the pendulum. Fig. 1 demonstrates that the amplitude decrease in the gravitational pendulum oscillations is accompanied by the amplitude increase in the spring pendulum oscillations. Thus, the total energy of the system remains unchanged with the friction absence. As a result, the graphs $\varphi(t)$ and $\Delta r(t)$ have the form of modulated functions that are similar to the effect of beating frequencies and oscillations of linearly coupled systems. Though, for two linear coupled oscillatory systems, the frequency of energy exchange between subsystems is determined by the system parameters and does not depend on the initial conditions. In the case of an elastic gravitational pendulum, the period of energy exchange can be changed in a wide range by the initial conditions. For example, a decrease in the initial amplitudes of oscillations leads to an increase in the period of energy transfer (Fig. 1).

Studies have shown that the value of the transferred energy between the coupled subsystems (ΔE) at the small initial deviations is determined by the equation

$$\Delta E=\left|E_0^g-2E_0^{sp}\right|\,, \qquad (5)$$

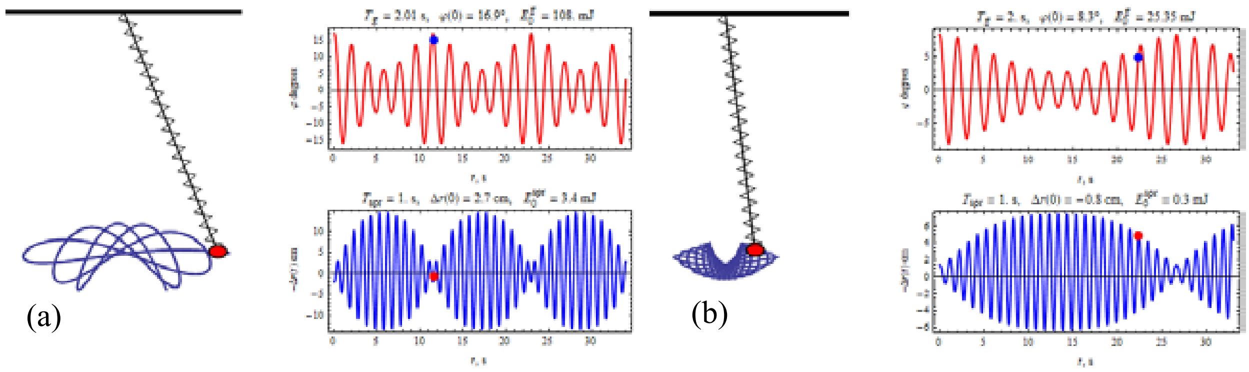

Fig. 1. Oscillations of a resonant pendulum: a) φ(0)=16.9º, $\Delta r(0) = 2.7cm$; b) φ(0)=8.3º, $\Delta r(0) = -0.8cm$.

where $E_0^g = mg\left[1-\cos\varphi(0)\right]r(0)$, $E_0^{sp} = k[\Delta r(0)]^2/2$ – the energies of the gravitational and spring pendulums at the initial moment of time. As it follows from (5), there are such initial conditions in which there is no energy exchange between subsystems and oscillations of pendulums occur independently. Since, the spring pendulum initial energy can be provided both by tension ($\Delta r(0) > 0$) and compression ($\Delta r(0) < 0$) of the spring, two types of such oscillations are possible (**A** and **B**, Fig. 2).

When changing the initial deviations of the pendulum, there are various stable periodic trajectories of the pendulum oscillations, which are called modes. The existence of stable modes is explained by the fact that when the period of energy exchange becomes a multiple of the period of the gravitational pendulum oscillations ($T=N\cdot T_g$.), the system returns to the same state that was in the beginning, and hence the trajectory will repeat. For the mode depicted on Fig. 1a the period of energy exchange $T = 12c$ (T_g=2 c, N=6) and on Fig. 1b $T = 26c$ (T_g=2 c, N=13). Each of these modes can exist at different values of the initial conditions. In Fig. 3 solid lines show the initial conditions of several oscillation modes. The period of energy transmission between subsystems is constant along these oscillation modes lines. On the inserts of Fig. 3 one can see trajectories, that are typical for these modes.

The simplest oscillation modes are observed at high energies, since then the energy transfer period is small and therefore contains a small number of the gravitational pendulum oscillations (N).

The blue dashed lines in Fig. 3 show stable periodic oscillating states **A** and **B** of the pendulum. The lines of all modes cross one of the dashed lines

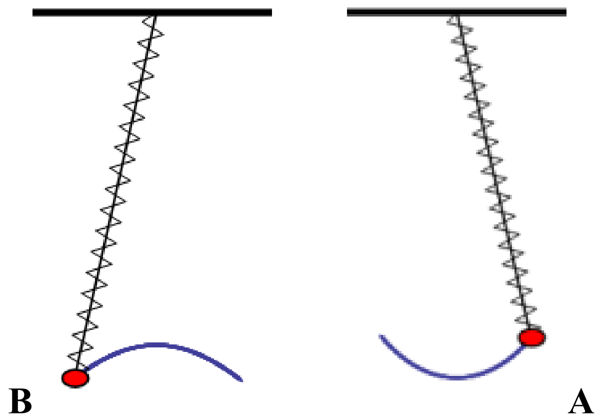

Fig. 2. Stable states of pendulum ($\Delta E = 0$)

A or **B**, so each oscillation mode can be in one of these states. Letter "a" or "b" in the name of the mode depict this fact. So, number N will denote the oscillation modes, and the Latin letters – their modifications. For example 2a, 2b, 3a, 3b,.... The Fig. 3 shows only the lines of the initial conditions of the pendulum, which correspond to the modes of oscillations with $N \le 6$. The trajectory of oscillating modes *N*a and *N*b are mirror symmetric to each other.

The mode lines of the type *N*a are monotonic functions but mode lines of the type *N*b are nonmonotonic functions. The lines of different modes characterized by the same number N converge asymptotically with the increase of the initial angle of the pendulum. Thus a region with a high density of lines is formed. The red dashed line in Fig. 3 divides the map of the pendulum's initial condition on the regions oscillation **A** and **B** types. The region near the red line is the region of initial conditions where the oscillations of the pendulum are unstable, and a small change in the initial deviations of the pendulum causes a large change in the

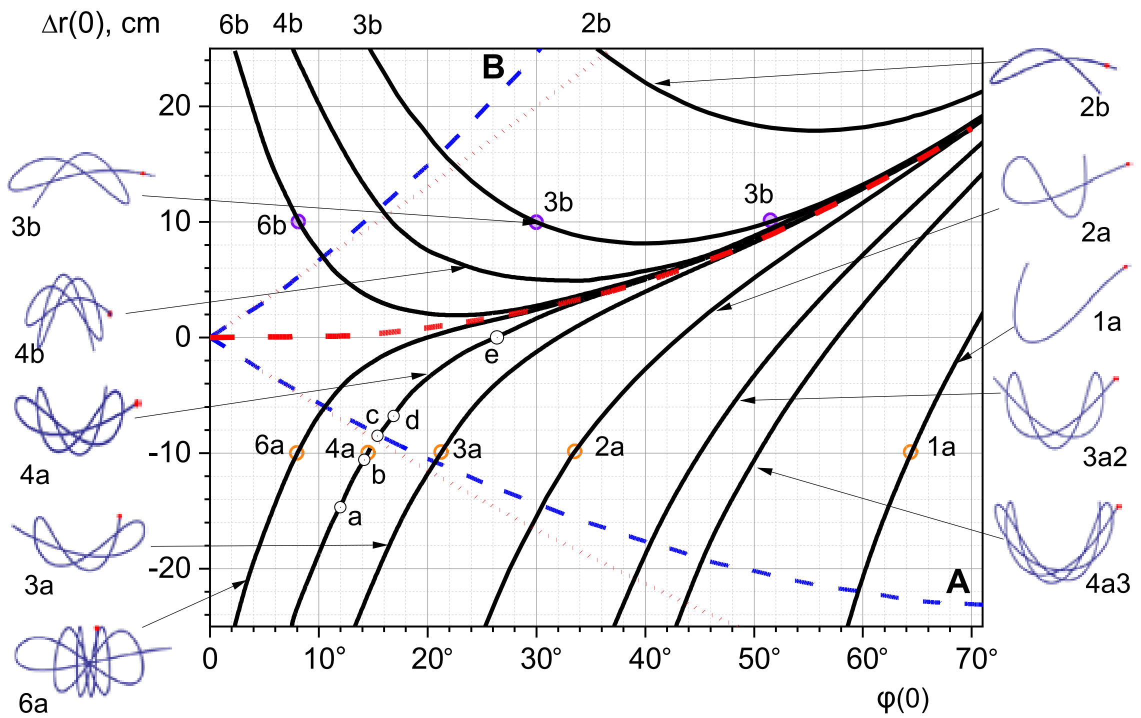

Fig. 3. Distribution of oscillating modes on the plane of the pendulum initial deviations – solid lines. States A and B – dashed lines. $E_0^g = 2E_0^{sp}$ – dotted lines

period of exchange energy and the trajectory of the pendulum.

Beside the described modes *N*a or *N*b there are such periodic modes of oscillation in which subsystems exchange energy several times until the pendulum returns to its original state. For example, in the mode 3a2 subsystems exchange energy twice, and in the mode 4a3 – thrice. The energy transfer period for these modes is equal to 3/2 T_g and 4/3·T_g, respectively.

The dependence energy transfer period on the initial angle φ(0) at the different fixed values of Δr(0) is shown in Fig. 4. The energy transfer period at $\Delta r(0) \le 0$ monotonically decreases as the initial angle of the pendulum increases. The dependence of the energy transfer period on the initial angle of the pendulum at $\Delta r(0) > 0$ has a non-monotonic form with a discontinuity at the point belonging to the bifurcation line.

In Fig. 5 on the example of mod 4a it is shown the peculiarities of the trajectories and parameters of pendulum oscillations when changing the initial conditions along the mode line (initial oscillation conditions, corresponding to points a, b, c, d, e, Fig. 3). For points b and d of 4a mode not only energy exchange and full energy are equal (Fig. 5b and Fig. 5d) but also pendulum movements have

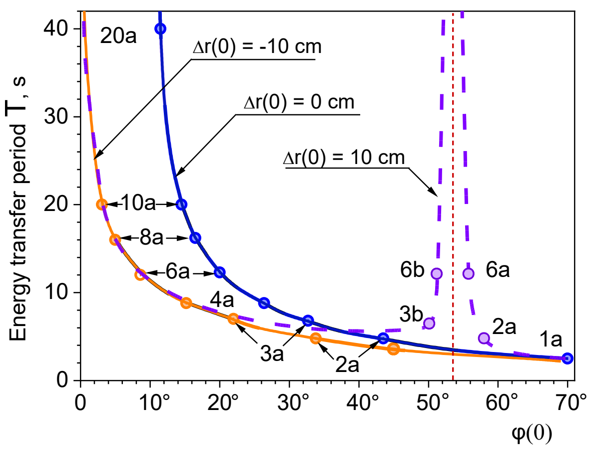

Fig. 4. Dependences energy transfer period on initial angle φ (0) at $\Delta r(0) = -10$; 0 ; 10 *cm*

the same trajectories. The difference is only in the initial phase of oscillation. In the first case, at the initial moment of time, the spring pendulum energy is transmitted to the gravitational one, whereas in the second case, it is vice versa. Ratio $E_0^{sp} / E_0^g = 2$ in (3) is a consequence of frequencies ratio. But Fig. 3 shows this ratio is performed for states A and B only at small amplitude of the gravitational pendulum (φ <15º). This is due to the dependence of gravitation pendulum frequency on an amplitude of oscillations and due to the approximate division

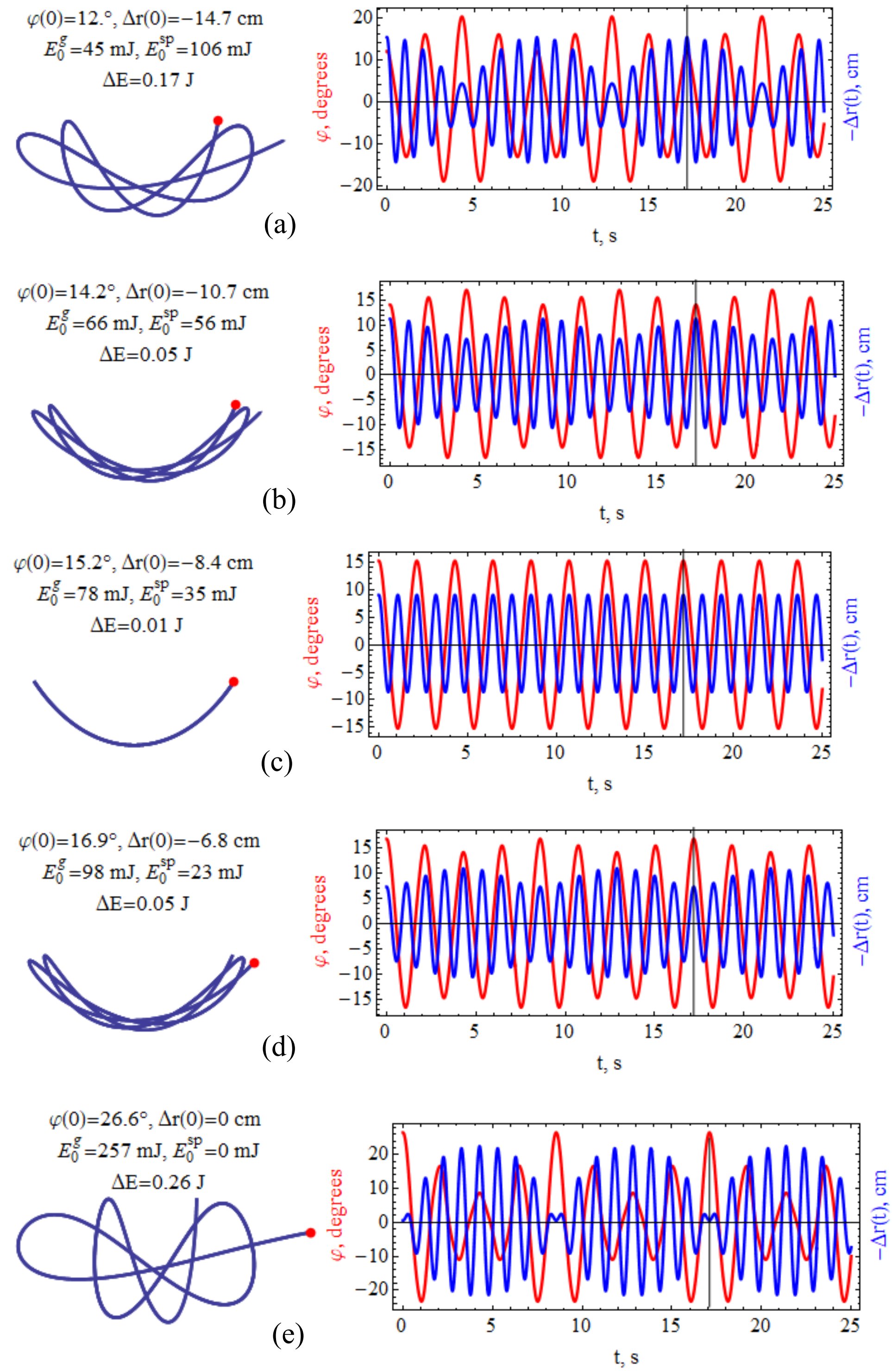

Fig. 5. Mode 4a at different initial conditions (depicted on the Fig. 3 by circles a, b, c, d, e)

of the full energy into the energies of the subsystems. There is a part of the energy that belongs to both subsystems, which could be called the interaction energy (Anisin, Davidovic, Babovic, 1993).

The phase trajectories of the pendulum oscillations are depicted on Fig. 6. The phase trajectories of pendulum oscillations in states **A** and **B** of 4a and 4b modes are similar to trajectories of non-interacting subsystems. The phase trajectory of spring oscillations with energy transfer (Fig. 6c) consist of several circles that are internally tangent.

SUMMARY

Thus, it was found some peculiarities of the elastic gravitational pendulum oscillations as the result of the motion equation numerical solution. Different oscillation modes are systematized and

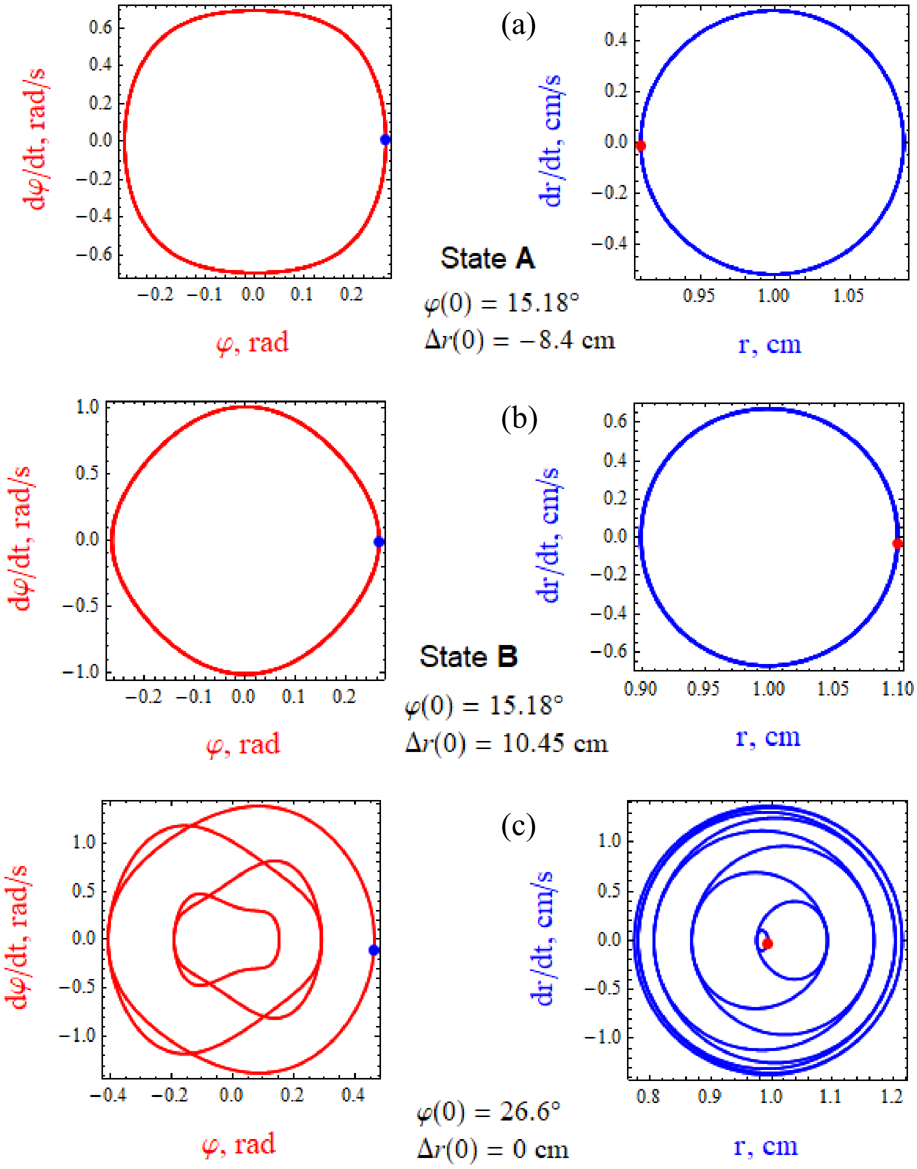

Fig. 6. Phase trajectories: (a) – 4a mode (state A), (b) – 4b mode (state B), (c) – 4b mode (with energy transfer)

their coupling with the initial conditions is established. The map of the initial conditions of oscillations of an elastic pendulum has been created, which allows to estimate the period of energy transfer and its magnitude based on the initial value of the tension and deflection angle of the pendulum. It was shown that for each oscillating mode a state (**A** or **B**) exists in which there is no energy exchange between subsystems that carry out independent harmonic oscillations.

It was shown that an increase in the total energy of a pendulum leads to a decrease in the period of energy exchange between subsystems.

The computer model is published with open code in Wolfram Language [10] and can be used in the educational process.

ЛІТЕРАТУРА:

1. Алдошин Г.Т. Яковлев С.П. Механическая модель колебательного спектра углекислого газа. *Журнал технической физики.* 2016. Т.86. № 12. С. 25–32.

2. Витт А., Горелик Г. Колебания упругого маятника как пример колебаний двух параметрически связанных линейных систем. *ЖТФ.* 1933. Т. III. Вып. 2–3. С. 294–307.

3. Olsson M. Why does a mass on a spring sometimes misbehave Am. J. Phys. 1976. 44. 1211-1212.

4. Anisin B., Davidovic D., Babovic V., Anisin B. On the linear theory of the elastic pendulum Eur. J. Phys. 1993. 14, 132-135.
5. Christensen J. An improved calculation of the mass for the resonant spring pendulum Am J. Phys. 2004. 72, 818.
6. Lai H. M. On recurrence phenomenon of a resonant spring pendulum Am. J. Phys. 1984. 52, 219-223.
7. Carretero-Gonzalez R., Nunez-Yepez N., Salas-Brito A. Regular and chaotic behavior in an extensible pendulum Eur. J. Phys. 1994. 15, 139-148.
8. Sousa M. C., Marcus F. A., Caldas I. L., Viana R. L. Energy distribution in intrinsically coupled systems: The spring pendulum paradigm Physica A. 2018. 509. 1110
9. Cross R. Experimental investigation of an elastic pendulum Eur. J. Phys. 2017. 38, 065004 (9pp).
10. Holovatsky V., Holovatska Y. Oscillations of an elastic pendulum. 2019. URL: http://demonstrations.wolfram.com/OscillationsOfAnElasticPendulum/

REFERENCES:

1. Aldoshin G T, Yakovlev S P (2015) Analytic Model of Vibrations of a Carbon Dioxide Molecule. Fermi Resonance Mechanics of Solids, 50 33-43.
2. Vitt A, Gorelik G (1933) Oscillations of an elastic pendulum as an example of the oscillations of two parametrically coupled linear systems Zhurnal Tekhnicheskoy Fiziki, 3 294-307.
3. Olsson M 1976 Why does a mass on a spring sometimes misbehave Am. J. Phys. 44 1211-1212.
4. Anisin B., Davidovic D., Babovic V., Anisin B. (1993) On the linear theory of the elastic pendulum Eur. J. Phys. 14 132-135.
5. Christensen J 2004 An improved calculation of the mass for the resonant spring pendulum Am J. Phys. 72 818.
6. Lai H M 1984 On recurrence phenomenon of a resonant spring pendulum Am. J. Phys. 52 219-223.
7. Carretero-Gonzalez R, Nunez-Yepez N, Salas-Brito A 1994 Regular and chaotic behavior in an extensible pendulum Eur. J. Phys. 15 139-148.
8. Sousa M C, Marcus F A, Caldas I L, Viana R L 2018 Energy distribution in intrinsically coupled systems: The spring pendulum paradigm Physica A 509 1110
9. Cross R. 2017 Experimental investigation of an elastic pendulum Eur. J. Phys. 38 065004 (9pp).
10. Holovatsky V., Holovatska Y. 2019. Oscillations of an elastic pendulum. URL: http://demonstrations.wolfram.com/OscillationsOfAnElasticPendulum/