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SOME PROBLEMS OF WHITE DWARF MODELLING

Main peculiarities of formulation main principles and criteria of white dwarf theories are analyzed. Short comparative analysis of main methods of modeling is represented. All these methods are based on the conditions of equilibrium. The role of A. Eddington, R. Fowler, E. Stoner and S. Chandrasekar researches in the creation this theory is discussed. A. Eddington proposed to use the Lane-Emden equations to construct the theory of white dwarfs, which allow us to describe processes in polytropic gas spheres. It is shown that Stoner method, Lane-Emden equations and Einstein equations is based on the search of equilbrium comditions for sphere or semple to sphere symmetries. The role of the development of theoretical physics (Fermi-Dirac statistics) in the creation of this theory is shown. It should be noted that thanks to Stoner's research, the Pauli principle and one of the first applications of Fermi-Dirac statistics for degenerate electronic systems appeared in the Bohr theory of the atom precisely in the theory of white dwarfs. Main peculiarities of Stoner method and Lane-Emden equtions are observed. Stoner's method is based on the idea of studying the equilibrium of a star based on energetic considerations. The Lane-Emden equations were constructed for gaseous spheres with different polytropic indices. It was Emden's introduction of thermodynamics into these equations that allowed them to be used in astrophysics. For these equations, it is necessary to additionally introduce the conditions for the rotation of the star. Einstein's equation is also on the one hand a condition of energy equilibrium (effective potential energy equals kinetic energy) for inhomogeneous systems, and on the other hand it is a generalization of the special theory of relativity to curvilinear geometry. The generalization of the interval itself is nothing more than a metric of the corresponding space-time. Rotation is included in the equation from the very beginning. Peculiarities of application methods of general relarivity for modelling white dwarf structure and processes are analyzed too.

***Key words:** critical processes, white dwarf, S. Chandrasrekhar, Lane-Emden equation, equilibrium conditions, phase transformarions, general relativity.*

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ДЕЯКІ ПРОБЛЕМИ МОДЕЛЮВАННЯ БІЛИХ КАРЛИКІВ

Проаналізовано основні особливості формулювання основних принципів і критеріїв теорій білих карликів. Подано короткий порівняльний аналіз основних методів моделювання. Всі ці методи базуються на умовах рівноваги. Обговорюється роль досліджень А. Еддінгтона, Р. Фаулера, Е. Стонера та С. Чандрасекара у створенні цієї теорії. А. Еддінгтон запропонував використовувати рівняння Лейна-Емдена для побудови теорії білих карликів, які дозволяють описувати процеси в політропних газових сферах. Показано, що метод Стонера, рівняння Лейна-Емдена та рівняння Ейнштейна ґрунтуються на пошуку умов рівноваги для симетрій сфери або відрізка до сфери. Показано роль розвитку теоретичної фізики (статистики Фермі-Дірака) у створенні цієї *теорії. Слід зазначити, що завдяки дослідженням Стонера в теорії атома Бора саме в теорії білих карликів з'явився принцип Паулі і одне з перших застосувань статистики Фермі-Дірака для вироджених електронних систем. Розглянуто основні особливості методу Стонера та рівнянь Лейна-Емдена. Метод Стонера базується на ідеї вивчення рівноваги зірки на основі енергетичних міркувань. Побудовано рівняння Лейна-Емдена для газоподібних куль з різними показниками політропії. Саме введення Емденом термодинаміки в ці рівняння дозволило використовувати їх в астрофізиці. Для цих рівнянь необхідно додатково ввести умови обертання зірки. Рівняння Ейнштейна також є, з одного боку, умовою енергетичної рівноваги (ефективна потенціальна енергія дорівнює кінетичній енергії) для неоднорідних систем, а з іншого – узагальненням спеціальної теорії відносності на криволінійну геометрію. Узагальнення самого інтервалу є не що інше, як метрика відповідного простору-часу. Обертання включено в рівняння з самого початку. Проаналізовано особливості застосування методів загальної теорії відносності для моделювання структури та процесів білих карликів.*

Ключові слова: *критичні процеси, білі карлики, С. Чандрасекар, рівняння Лейна-Емдена, умови рівноваги, фазові перетворення, загальна теорія відносності.*

Introduction

Main peculiarities of formulation main principles and criteria of white dwarf theories are analyzed. Short comparative analysis of main methods of modeling is represented. All thes methods must be based on the conditions of equilibrium. It is shown that Stoner method (Stoner, 1929), Lane-Emden equations (Chandrasekhar, 1938) and Einstein equations (Chandrasekhar, 1964) is based on the search of equilbrium conditions for sphere or sample to sphere symmetries. Main peculiarities of Stoner method and Lane-Emden equtions are observed. Peculiarities of application methods of general relarivityy for modelling white dwarf structure and processes are analyzed too.

Main concepts of theory of white dwarfs are discussing (Vavrukh, 2018). The role of A. Eddington (Eddington, 1926), R. Fowler (1926), E. Stoner (1929) and S. Chandrasekar (1930) researches in the creation this theory is discussed. The role of the development of theoretical physics (Fermi-Dirac statistics) in the creation of this theory is shown (Stoner, 1924; Fowler, 1926).

It should be noted that thanks to Stoner's research (Stoner, 1924), the Pauli principle and one of the first applications of Fermi-Dirac statistics for degenerate electronic systems appeared in the Bohr theory of the atom precisely in the theory of white dwarfs.

A. Eddington, R. Fowler, their student S. Chandrasekhar, E. Stoner, V. Anderson and others made the main contribution to the development of the theory of white dwarfs (Nauenberg, 2008; Vavrukh, 2018).

A. Eddington initiated the study of white dwarfs and, in addition, pointed out that the source of the stars' energy is thermonuclear reactions of hydrogen and helium synthesis (Eddington, 1926). He also proposed to use the Lane-Emden equations to construct the theory of white dwarfs, which allow us to describe processes in polytropic gas spheres. This method for white dwarfs was developed by A. Fowler (Fowler, 2029) and most of all by S. Chandrasekhar (Chandrasekhar, 1938).

R. Fowler was the first to draw attention to the use of Fermi-Dirac statistics for the theory of white dwarfs (Fawler, 1926). However, the first theory for former dwarfs was built by E. Stoner (Stoner, 1929). At the heart of his theory he put the variational principle, which he used for total energy. Vio estimated the density and mass of the white dwarf. This same method was used by S. Chandrasekhar to determine the density of a white dwarf. He built a more complete theory of a gray dwarf based on the Lane-Emden equations.

It should be noted that all astrophysical models are built based on considerations of equilibrium, and are still tied in one way or another to spherical symmetry. Thus, the Lane-Emden equations are derived for the equilibrium conditions of a gas sphere taking into account the corresponding polytropic process. Chandrasekhar separately derived this equation for isothermal case too (Chandrasekhar, 1938). However, these equations do not take into account the rotation of the star (Vavrukh, 2018). Einstein's equation is nothing but the equality of potential (effective) energy, which also takes into account the rotation and kinetic energy (Barrow, 2007). That is, it is nothing but an extension of the methods of celestial mechanics. All the metrics that are in the general theory of relativity, roughly speaking, are derived from the spherical metric (Barrow, 2007). Therefore this metod ass more universal for astrophysical applications was recommended by A. Eddingtom (Chandrasekhar, 1964).

In the theory of degenerate dwarfs by S. Chandrasekhar is generalized by constructing multiparameter and multiphase models that take into account the incomplete degeneracy of the electronic subsystem, the presence of interparticle interactions, magnetic fields, variable chemical content along the radius, and axial rotation of stars (Varukh, 2018). It is allowed to adequately describe and provide interpretation of modern observed data. Based on the solution of the equilibrium equations, the dependence of the structural and energy characteristics of dwarfs on the parameters of the models was determined. The inverse problem of the theory of degenerate dwarfs was solved – the determination of the main structural and thermodynamic parameters of the models based on data on the masses, radii, and effective temperatures of observed field dwarfs and dwarfs in binaries systems (Vavrukh, 2018).

S. Chandrasekhar theory of white dwarfs was developed in second half of 20th century. In the works of E. Shatsman, S. Chandrasekhar, S. Kaplan, R. James L. Mestell, I. Zeldovich, and I. Novikov, an elementary theory of cooling of degenerate dwarfs was constructed (Vavrukh, 2018). The issues of stability in relation to neutronization processes, the effects of the general theory of relativity and the influence of axial rotation, etc., were also considered. Much less attention has been paid to the generalization of models of the internal structure of dwarfs. Here it should be noted the work of E. Solpiter on the equation of state of the electron-nuclear model at high couplings with approximate consideration of Coulomb interactions (at T = 0 K) and the work of T. Hamada and E. Solpiter devoted to the calculation of the «mass-radius» ratio for homogeneous two-component models corresponding to chemical elements with a nuclear charge of $2 \le z \le 26$ based on the equation of state obtained by E. Solpiter and with the acceleration of neutronization processes, but taking into account the effects of the general theory of relativity (Vavrukh, 2018).

Main results

The existence of a mass limit for white dwarfs is usually attributed solely to the late astrophysicist Subrahmanyan Chandrasekhar, and this limit is named after him (Chandrasekhar, 1938). But as is often the case, the history of this discovery is more nuanced.In this note I will show that the existence of a maximum mass was first establishedby Edmund C. Stoner, a physicist who began experimental research under the supervision of Rutherford at the Cavendish in Cambridge, but later switched to theoretical work. Rutherford recommended Stoner to a position at the Physics department of the University of Leeds where he spent his entire career (Nauenberg, 2008).

According to G. Cantor, he was "probably the leading Cavendish-trained theoretical physicist of the 1920's, although he learned theory mostly on his own, and became known for his work on magnetism (Nauenberg, 2008). Unfortunately, Stoner suffered from diabetes and poor health which restricted his travels, and this may account for the fact that he did not receive wider recognition for his achievements. In 1924 Stoner wrote a paper on the distribution of electrons among atomic levels (Stoner, 1924). In the preface of the fourth edition of his classic book, "Atomic Structure and Spectral Lines", Arnold Sommerfeld gave special mention to " einen grossen Fortschritt [a great advancement]" brought about by Stoner's analysis, which then came to the attention of Wolfgang Pauli, and played and important role in his formulation of the exclusion principle in quantum physics (Nauenberg, 2008). Therefore, it is not surprising that Stoner's interest in white dwarfs was aroused by Ralph H. Fowler's suggestion (Fowler, 1926) that the exclusion principle could be applied to solve a major puzzle, the origin of the extreme high density of white dwarfs (Nauenberg, 2008), which could not be explained by classical physics.

At the time, the conventional wisdom was that the source of internal pressure which maintained all stars in equilibrium against gravitational collapse was the internal pressure of the matter composing the star which had been heated into a gas presumably, according to Eddington, by "subatomic energy'' (Eddington, 1926). But when this supply of energy is exhausted and the star cools, Fowler proposed that a new equilibrium would ensue, even at zero temperature, due to the "degeneracy" pressure of the electrons caused by the exclusion principle (Nauenberg, 2008). Fowler, however, did not attempt to determine the equilibrium properties of such a star which he regarded as "strictly analogous to one giant molecule in the ground state''. Apparently he was unaware that at the time, Llewellyn H. Thomas had developed a mathematical method to solve this problem in atomic physics (Nauenberg, 2008). Subsequently, Stoner developed a novel minimum energy principle to obtain the equilibrium properties of such dense stars, and by applying Fowler's non-relativistic equation of state for a degenerate electron gas in a constant density approximation, he found that the density increases with the square of the mass of the star (Nauenberg, 2008). In such a gas the mean momentum of an electron is proportional to the cube root of the density, and Wilhem Anderson, a privatdozent at Tartu University, Estonia, who had read Stoner's paper, noticed that for the mass of a white dwarf comparable to or higher than the mass of the Sun, the density calculated from Stoner's non-relativistic mass-density relation implied that the electrons become relativistic (Nauenberg, 2008). Hence, Anderson concluded that in this regime, this relation gave "gröblich falschen Resultaten [gross false results]" for the properties of a white dwarf. He attemped to extend the equation of state of a degenerate electron gas to the relativistic domain, but he gave an incorrect formulation which, fortuitously, indicated that Stoner's minimum energy principle implied a maximum value for the white dwarf mass. Alerted by Anderson's paper, Stoner then derived the correct relativistic equation of state16, and re-calculated, in a constant density approximation, the properties of white dwarfs for arbitrary densities (Nauenberg, 2008). Thus, he obtained, now on solid theoretical grounds, the surprising result that when the density approaches infinity, the mass of the star reaches a maximum value.

Stoner's method (Nauenberg, 2008) for obtaining the properties of white dwarfs was basedon his concept that at equilibrium, the sum of the internal energy and the gravitational energy of the star should be a minimum for a fixed mass of the star.

Fowler had assumed that the atoms in a white dwarf were completely ionized, and that the internal energy and pressure was entirely due to a degenerate electron gas, while the ions mainly accounted for the mass of the star. Stoner understood that as the star contracts, the gravitational energy decreases, and since the density increases, the internal energy also increases. Hence, the total energy of the star either decreases or increases during the contraction of the star. By conservation of energy, when the total energy of the star decreases, radiation and/or other forms of energy must be emitted by the star. But without an external source of energy, the total energy of an isolated star cannot increase. Hence the contraction of the star must end if the total energy reaches a minimum, and then he star reaches an equilibrium 1

Stoner's method for obtaining the properties of white dwarfs was based on his concept that at equilibrium, the sum of the internal energy and the gravitational energy of the star should be a minimum for a fixed mass of the star. Let E_G be the gravitational energy of the idealized star, E_K the total kinetic energy of rlectrons, n the number of electrons per unite volume. Then (Nauenberg, 2008) the equilibrium comdition is given by

$$\frac{d}{dn}\left(E_G + E_K\right) = 0. \tag{1}$$

The total kinetic energy of electrons is equaled according to (Nauenberg, 2008):

$$E_K = \frac{8\pi V m_0^4 c^5}{h^3}\left[\frac{1}{8}x\left(1+x^2\right)^{1/2}\left(1+2x^2\right) - \frac{1}{3}x^3 - \log\left\{x+\left(1+x^2\right)^{1/2}\right\}\right], \tag{2}$$

where

$$f(x) = \frac{1}{8}\left[x\left(1+x^2\right)^{1/2}\left(1+2x^2\right) - \log\left\{x+\left(1+x^2\right)^{1/2}\right\}\right], \tag{3}$$

with

$$x = \frac{p_0}{m_0 c} = \frac{1}{m_0 c}\left(\frac{3h^3 n}{8\pi}\right)^{1/3}, \tag{4}$$

and V is total volume of electrons, p_0 is maximum momentum.

For the gravitational potential energy with 2.5 m_H as the mean molecular weight of the material of the star (Nauenberg, 2008),

$$E_G = 3^{1/3}\left(\frac{4\pi}{5}\right)^{2/3}\frac{1}{h}GM^{5/3}m_H^{1/3}m_0 cx. \tag{5}$$

Kinetic energy (2) may be rewritten as

$$E_K = \frac{8\pi V m_0^4 c^5}{h^3} f_1(x), \tag{6}$$

where

$$f_1(x) = \left[\frac{1}{8}x\left(1+x^2\right)^{1/2}\left(1+2x^2\right) - \frac{1}{3}x^3 - \log\left\{x+\left(1+x^2\right)^{1/2}\right\}\right]. \tag{7}$$

Substituting $M/2.5m_H n$ for V, and further substituting for n as above,

$$E_K = \frac{3Mm_0c^2}{2.5m_H}\frac{f_1(x)}{x^3}. \tag{8}$$

Substituting receiving values of E_G and E_K in equilibrium condition we have

$$\frac{d}{dx}\left(\frac{f_1(x)}{x^3}\right) = 10^{1/3}\left(\frac{\pi}{3}\right)^{2/3}\frac{Gm_H^{4/3}M^{2/3}}{hc}. \tag{9}$$

Inserting numerical values,

$$\frac{d}{dx}\left(\frac{f_1(x)}{x^3}\right)=F(x)=1.483\cdot10^{-23}M^{2/3}, \qquad (10)$$

where do we have

$$M=1.751\cdot10^{34}\left[F(x)\right]^{3/2}. \qquad (11)$$

Since the mean molecular weight is about $2.5m_H$ the limitmg demsity is given by

$$\rho_0=2.5m_H n=4.15\cdot10^{-24}n. \qquad (12)$$

The density of the white dwarf stars is reconsidered from the point of view of the theory of the polytropic gas spheres and gives for the mean density of a white dwarf (under ideal conditions) the formula (Chandrasekhar, 1931)

$$\rho_{Ch}=2.162\cdot10^{6}\cdot\left(M/M_G\right)^2. \qquad (13)$$

Corresponding Stoner formula has next form (Stoner, 1929)

$$\rho_S=3.977\cdot10^{6}\cdot\left(M/M_G\right)^2. \qquad (14)$$

As we see $\rho_S/\rho_{Ch}=1.84$ (Nauenberg, 2008).

It should be noted that Chandrasekhar in this case used the same method as Stoner (Chandrasekhar,1938).

Fowler had assumed that the atoms in a white dwarf were completely ionized, and that the internal energy and pressure was entirely due to a degenerate electron gas, while the ions mainly accounted for the mass of the star. Stoner understood that as the star contracts, the gravitational energy decreases, and since the density increases, the internal energy also increases. Hence, the total energy of the star either decreases or increases during the contraction of the star. By conservation of energy, when the total energy of the star decreases, radiation and/or other forms of energy must be emitted by the star. But without an external source of energy, the total energy of an isolated star cannot increase. Hence the contraction of the star must end if the total energy reaches a minimum, and then the star reaches an equilibrium (Nauenberg, 2008).

To calculate the density at which the total energy minimum occurs, Stoner started with an approximation by assuming that the density was uniform. In his first paper (Nauenberg, 2008) he applied Fowler's non-relativistic form for the degeneracy energy, and he found that the density depends quadratically on the mass of the star. Later, in collaboration with F. Tyler (Nauenberg, 2008), he also considered the modification for non-relativistic degeneracy when the density varies according to a polytrope distribution with index $n = 3/2$.Then, after Anderson (Nauenberg, 2008) pointed out that for a white dwarf with a mass of the order of the mass of the Sun Stoner's analysis implied that the electrons become relativistic, Stoner obtained the general relativistic equation of state for a degenerate electron gas, and he applied it to obtain the mass-density relation of white dwarfs for arbitrary densities (Nauenberg, 2008). By means of his minimum energy principle, he obtained and analytic expression which gave this relation in parametric form, showing that the density is a function that increases monotonically, and more rapidly than the square of the star's mass. In particular, he obtained the fundamental result that the density approaches infinity for a finite mass. This is the celebrated limiting mass of white dwarfs, in which the mass scale is entirely determined by some of the fundamental constants of Nature.

Chandrasekhar's early method was based on applying the Lane-Emden polytrope solution of the differential equation for gravitational equilibrium for the equation of state of a degenerate electron gas which obey power laws in the non-relativistic and the extreme relativistic regime (Chandrasekhar, 1938). He obtained results similar to Stoner's for the white dwarf mass-density relation in the non-relativistic regime, and for the critical white dwarf mass in the extreme relativistic regime (Nauenberg, 2008). For a power law dependence of the pressure p on the density ρ, i.e. $p\sim\rho^{\gamma}$, where the exponent γ is a constant, the solution of this equation is given by the Lane-Emden polytrope of index n, where $\gamma=1+\frac{1}{n}$. Chandrasekhar found these solutions in Eddington's book, "The Internal Constitution of Stars" (Eddington, 1926), which also contained the relations and numerical quantities that he needed for his calculations. In the non-relativistic limit, $\gamma = 5/3$, corresponding to a polytrope with index $n = 3/2$, and this Lane-Emden solution gives the central or mean density dependence as the square of the mass of the star, the same result which Stoner had obtained two years earlier in the uniform density approximation. Substituting Fowler's non-relativistic pressure density relation, Chandrasekhar found that the magnitude of this dependence is smaller than Stoner's value by a factor approximately equal to two (Nauenberg, 2008). But somewhat earlier, motivated by Stoner's work, E. Milne already had carried out this calculation (Nauenberg, 2008), and at about the same time Stoner and Tyler also had applied the n = 3/2 polytrope density in Stoner's minimum energy method, and obtained the same result (Nauenberg, 2008). In the extreme relativistic limit, γ = 4/3, the corresponding polytrope has index $n = 3$, and the mass is independent of the central or mean density of the star. Thus Chandrasekhar calculated the magnitude of the critical mass of white dwarfs, which depends on the fundamental constants of nature, as had been shown a year before by Stoner, and on a dimensional constant for the n = 3 polytrope. This gave a critical mass about 20 % smaller than Stoner's value for the uniform density approximation (Nauenberg, 2008). By his own admission, however, Chandrasekhar was puzzled by his result [21], and he was not able to show until several months later that the critical mass was a maximum, and that in this limit the density was infinite. Moreover (Nauenberg, 2008), he did not pursue the implications of this result, and for several years he assumed that at a certain value of the density, matter would become incompressible, an idea proposed earlier by Milne to avoid infinite density at the center of his models of a star. Chandrasekhar formulated this idea as follows":»We are bound to assume therefore that a stage must come beyond which the equation of state $p = K\rho^{4/3}$ is not valid, for otherwise we are led to the physically inconceivable result that for $M = 0.92M_s$ [M_s=solar mass and μ = 2.5], r_1 = 0, and $\rho = \infty$. As we do not know physically what the equation of state is thatwe are to take, we assume for definiteness the equation for the homogeneous material $\rho = \rho_{max}$, where ρ_{max} is the maximum density of which the material is capable…» (Nauenberg, 2008).

It is of interest to inquire what the relation is between Stoner's minimum energy method and Chandrasekhar's equation of gravitational equation. Treating Stoner's minimum energy principle as a variational problem in which the total energy is a functional of the density, and this density is a variable function of the radial density, this variational approach leads to the quantum mechanical ground state of an electron gas in the gravitational field of the ions, which maintain charge neutrality. This connection explains why Stoner and Chandrasekhar obtained the same relations for the density and mass of the star as functions of fundamental constants, but with somewhat different dimensionless quantities. In particular, I will show that the solution to the generalized form of Stoner's variational problem for the minimum of the total energy of a dense star leads to the differential equation of gravitational equilibrium which Chandrasekhar applied in his work. I have not found any evidence, however, that either Stoner or Chandrasekhar were aware of this connection.

The total energy E of a zero temperature dense star supported entirely by degeneracy pressure against the gravitational attractive forces can be written as a functional of the mass density distribution ρ integrated over the volume of the star,

$$E = \int dv \left[\varepsilon(\rho) - u(p,r) \right], \tag{15}$$

where $\varepsilon(\rho)$ is the internal energy given as a function of of the mass density ρ by Stoner' relativistic equation of state for a electron degenerate gas, $u(p,r)$ is gravitational energy

$$u(p,r) = -\frac{1}{2} G \int dv' \frac{\rho(r')\rho(r)}{|r - r'|}, \tag{16}$$

and G is Newton's gravity constant.

The equilibrium distribution ρ as a function of position r can be determined by evaluating the minimum of E subject to the condition that the total mass $M = \int dv\rho$ is fixed (Nauenberg, 2008). Assuming that ρ depends only on the radial distance r from the center of the star, this variational problem leads to the differential equation for gravitational equilibrium,

$$\frac{dP}{dr} = -G \frac{M(r)\rho(r)}{r^2} \tag{17}$$

here $P = \rho d\varepsilon/d\rho - \varepsilon$ is the pressure, and $M(r) = 4\pi \int r^2 \frac{M(r)\rho(r)}{r^2} dr$ is the mass inside the radius r. In the uniform density approximation, the solution of Stoner's minimum energy principle gives the relation

$$P = (3/20\,\pi)GM^2/R^4, \tag{18}$$

where P is the mean pressure, M is the mass and R is the radius of the star. Stoner's relativistic equation of state for the pressure – density relation of a degenerate electron gas was first given in the form $P = Ax^4F(x)$, where

$$F(x) = \frac{1}{8x^3}\left[\frac{3}{x} \log\left(x + \sqrt{1 + x^2} \right) + \sqrt{1 + x^2}\left(2x^3 - 3 \right) \right], \tag{19}$$

and $x = \frac{3nh}{8\pi mc}$, Here n is the electrton density $n = \frac{3M}{4\pi R^3 m_H \mu}$, m_H is the proton mass, h is Planck's constant, c is the velocity of light, μ is the molecular weight and $\frac{8\pi m^4 c^5}{3h^3}$. Hence Stoner's analytic solution for the mass M of a a white dwarf takes the form $M = M_c\left(4F(x)\right)^{3/2}$. In the limit of small density $x = 1$, $F(x) = x/5$, and $P = (1/20)(3/\pi)^{2/3}(h^2/m)n^{5/3}$, which corresponds to Fowler's result for the pressure-density relation in the non-relativistic limit. In this limit we recover Stoner's original relation that the density n is proportional to the square of the mass M of the star, $n = (10\pi/3)(mc/h)^3 (M/M_s)^2$. The maximum momentum of the electrons is $p = (mc)x$, and therefore when x is of order one or larger the effects of relativity become important, as was first pointed out by Anderson, and independently by Chandrasekhar. In the limit of infinite density, $x \to \infty$, $F(x) \to$ ¼,, which gives $P = (1/8)(3/\pi)^{1/3} n^{4/3}$, and $M = M_c$, with Stoner's critical mass expressed in terms of some of the fundamental constants of nature (Nauenberg, 2008),

$$M_{cS} = \left(3/16\pi\right)\left(5hc/2G\right)^{3/2}\left(m_H\mu\right)^{-2}. \qquad (20)$$

Chandrasekhar's result for the critical mass, expressed in terms of fundamental constants, corresponds

$$M_{cCh} = u\left(\sqrt{6}/8\pi\right)\left(hc/G\right)^{3/2}\left(m_H\mu\right)^{-2}, \qquad (21)$$

where $u = 2.018\ldots$ is a constant obtained bynumerically integrating the equation of gravitational equilibrium for an $n = 3$ polytrope. It can be readily verified that the critical mass evaluated with a mass density distribution corresponding to an $n = 3$ polytrope is 20% smaller than for a uniform density distribution. In other words

$$\frac{M_{cS}}{M_{cCh}} = 1.2. \qquad (22)$$

Astrophysical methods used to study the structure of stars are characterized by the use of physical assumptions, the choice of which is dictated solely by the convenience of calculations (Chandrasekhar, 1964).

The Chandrasekhar limit is the maximum mass of a stable white dwarf star (Chandrasekhar, 1938). The currently accepted value of the Chandrasekhar limit is about 1.4 $M_\odot$ (2.765×10^{30} kg).

White dwarfs resist gravitational collapse primarily through electron degeneracy pressure, compared to main sequence stars, which resist collapse through thermal pressure. The Chandrasekhar limit is the mass above which electron degeneracy pressure in the star's core is insufficient to balance the star's own gravitational self-attraction.

Normal stars fuse gravitationally compressed hydrogen into helium, generating vast amounts of heat. As the hydrogen is consumed, the stars' core compresses further allowing the helium and heavier nuclei to fuse ultimately resulting in stable iron nuclei, a process called stellar evolution. The next step depends upon the mass of the star. Stars below the Chandrasekhar limit become stable white dwarf stars, remaining that way throughout the rest of the history of the universe absent external forces. Stars above the limit can become neutron stars or black holes.

The Chandrasekhar limit is a consequence of competition between gravity and electron degeneracy pressure. Electron degeneracy pressure is a quantum-mechanical effect arising from the Pauli exclusion principle. Since electrons are fermions, no two electrons can be in the same state, so not all electrons can be in the minimum-energy level. Rather, electrons must occupy a band of energy levels. Compression of the electron gas increases the number of electrons in a given volume and raises the maximum energy level in the occupied band. Therefore, the energy of the electrons increases on compression, so pressure must be exerted on the electron gas to compress it, producing electron degeneracy pressure. With sufficient compression, electrons are forced into nuclei in the process of electron capture, relieving the pressure (Nauenberg, 2008).

In the nonrelativistic case, electron degeneracy pressure gives rise to an equation of state of the form $p = K_1\rho^{5/3}$, where P is the pressure, ρ is the mass density, and K_1 is a constant. Solving the hydrostatic equation leads to a model white dwarf that is a polytrope of index 3/2 – and therefore has radius inversely proportional to the cube root of its mass, and volume inversely proportional to its mass (Nauenberg, 2008). As the mass of a model white dwarf increases, the typical energies to which degeneracy pressure forces the electrons are no longer negligible relative to their rest masses. The velocities of the electrons approach the speed of light, and special relativity must be taken into

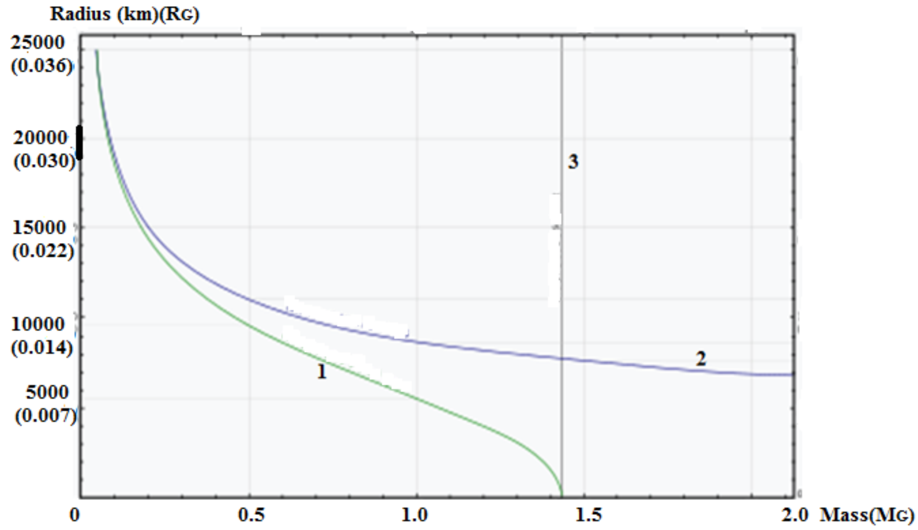

Fig. 1. Radius–mass relations for a model white dwarf [20-22]. 1 – Using the general pressure law for an ideal Fermi gas; 2 – Non-relativistic ideal Fermi gas; 3 – Ultrarelativistic limit (Nauenberg, 2008)

account. In the strongly relativistic limit, the equation of state takes the form $P = K_2\rho^{4/3}$. This yields a polytrope of index 3, which has a total mass, M_{limit}, depending only on K_2 (Nauenberg, 2008).

For a fully relativistic treatment, the equation of state used interpolates between the equations $P = K_1\rho^{5/3}$ for small ρ and $P = K_2\rho^{4/3}$ for large ρ. When this is done, the model radius still decreases with mass, but becomes zero at M_{limit}. This is the Chandrasekhar limit (Nauenberg, 2008).The curves of radius against mass for the non-relativistic and relativistic models are shown in the Fig. 1 (Nauenberg, 2008). They are colored blue and green, respectively. μ_e has been set equal to 2. Radius is measured in standard solar radii or kilometers, and mass in standard solar masses.

For a more detailed analysis of astrophysical and cosmological processes, it is worth using Einstein's equations, or their simplified version of the Friedmann equation. Since they are practically a generalization of the energy balance equations (potential energy is equal to kinetic energy) (Barrow, 2007). The connection with theormodynamics of these equations was established by Tolman (Tolman, 1931). The current state of this problem is given in Danylchenko (Danylchenko, 2022). If in the Lane-Emden equations physical processes are introduced through the polytrope, then in the general theory of relativity through the curvature of space-time (Chandrasekhar, 1964).

In addition, these processes of the transition of stars from one state to another are the subject of the physics of critical phenomena and from this point of view, more universal methods should be sought for the construction of such theories and models (Trokhimchuck, 2024).

Conclusions

1. Main peculiarities of modeling the dwite stars are represented.
2. Comparative analysis of Stoner energetic method and Lane-Emden equation method is observed.
3. It is shown that general relativity method is more general as Stoner and Lane-Emden method.
4. Perspective of the investigations new methods of modeling the white dwarfs and other compact astrophysical objects as development pf physics of critical processes are discussed too.

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